

# ON GENERAL CLOSURE OPERATORS AND QUASI FACTORIZATION STRUCTURES

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**ABSTRACT.** In this article the notions of (quasi weakly hereditary) general closure operator  $\mathbf{C}$  on a category  $\mathcal{X}$  with respect to a class  $\mathcal{M}$  of morphisms, and quasi factorization structures in a category  $\mathcal{X}$  are introduced. It is shown that under certain conditions, if  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure in  $\mathcal{X}$ , then  $\mathcal{X}$  has quasi right  $\mathcal{M}$ -factorization structure and quasi left  $\mathcal{E}$ -factorization structure. It is also shown that for a quasi weakly hereditary and quasi idempotent QCD-closure operator with respect to a certain class  $\mathcal{M}$ , every quasi factorization structure  $(\mathcal{E}, \mathcal{M})$  yields a quasi factorization structure relative to the given closure operator; and that for a closure operator with respect to a certain class  $\mathcal{M}$ , if the pair of classes of quasi dense and quasi closed morphisms forms a quasi factorization structure, then the closure operator is both quasi weakly hereditary and quasi idempotent. Several illustrative examples are furnished.

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## 1. INTRODUCTION

Closure operators have been around for almost one century in the context of categories of topological spaces and lattices. In [15], Salbany introduces a particular closure operator in the category of topological spaces. This idea was later transformed to an arbitrary category, which led to the general concept of categorical closure operators, [6], [4], [5]. Weakly hereditary and idempotent closure operators play an important role, as they arise from factorization structures. In [13], quasi right factorization structures are introduced and their connection with closure operators are investigated, while quasi left factorization structures appear in [10].

In this article, the connections between quasi right factorization structures, quasi left factorization structures, quasi factorization structures and closure operators are further investigated. In section 1, the notions of quasi monomorphism and quasi epimorphism and some preliminary results are given. In section 2, the definition of a general closure operator on a category  $\mathcal{X}$  with respect to the class  $\mathcal{M}$  of morphisms is introduced, some related results and several examples are also given. In section 3 after defining quasi weakly hereditary closure operator, we prove that for a quasi idempotent closure operator we have a quasi right factorization structure and for a quasi weakly hereditary closure operator under some conditions we have a quasi left factorization structure. In section 4, for morphism classes  $\mathcal{E}$  and  $\mathcal{M}$ , the notion of  $(\mathcal{E}, \mathcal{M})$ -quasi factorization structure is introduced and examples of quasi factorization structures that are not weak factorization structures are furnished. It is shown that if  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure in  $\mathcal{X}$ , then  $\mathcal{X}$  has quasi right  $\mathcal{M}$ -factorization structure provided that  $\mathcal{M}$  has  $\mathcal{X}$ -pullbacks and it has quasi left  $\mathcal{E}$ -factorization structure provided that  $\mathcal{M} \subseteq \text{Mon}(\mathcal{X})$ , the class of monomorphisms, and  $\mathcal{E}$  has  $\mathcal{X}$ -pushout. It is also shown that for a quasi weakly hereditary and quasi idempotent QCD-closure operator with respect to a class  $\mathcal{M}$  that is contained in the class of quasi monomorphisms and is closed under composition, every quasi factorization structure  $(\mathcal{E}, \mathcal{M})$  yields a quasi factorization structure relative to the given closure operator. Finally it is proved that for a closure operator with respect to a class  $\mathcal{M}$  that is contained in the class of strongly quasi monomorphisms and is a codoman, if the pair of classes of quasi dense and quasi closed morphisms forms a quasi factorization structure, then the closure operator is both quasi weakly hereditary and quasi idempotent.

To this end we will give some basic definitions and results which will be used in the following sections.

**Definition 1.1.** [13]. Let  $\mathcal{M}$  be a class of morphisms in  $\mathcal{X}$ . We say that  $\mathcal{X}$  has *quasi right  $\mathcal{M}$ -factorizations* or  $\mathcal{M}$  is a *quasi right factorization structure* in  $\mathcal{X}$ , whenever for all morphisms  $Y \xrightarrow{f} X$  in  $\mathcal{X}$ , there exists

$M \xrightarrow{m_f} X \in \mathcal{M}/X$  such that:

- (a)  $f = m_f g$  for some  $g$ ;
- (b) if there exists  $m \in \mathcal{M}/X$  such that  $f = mg$  for some  $g$ , then  $m_f = mh$  for some  $h$ .

$m_f$  is called a quasi right part of  $f$ .

With  $\langle m \rangle$  denoting the sieve generated by  $m$ , (a) is equivalent to:

- (a')  $\langle f \rangle \subseteq \langle m_f \rangle$ ;

and (b) is equivalent to:

- (b') if there exists  $m \in \mathcal{M}/X$  such that  $\langle f \rangle \subseteq \langle m \rangle$ , then  $\langle m_f \rangle \subseteq \langle m \rangle$ .

Note that right  $\mathcal{M}$ -factorizations as defined in [4] are quasi right  $\mathcal{M}$ -factorizations.

**Lemma 1.2.** [13]. Suppose  $\mathcal{X}$  has quasi right  $\mathcal{M}$ -factorizations. Let  $f$  be a morphism in  $\mathcal{X}$  and  $m_f$  be a quasi right part of  $f$ .

- (a) If  $f \in \mathcal{M}$ , then  $\langle m_f \rangle = \langle f \rangle$ .
- (b)  $m$  is a quasi right part of  $f$  if and only if  $m \in \mathcal{M}$  and  $\langle m \rangle = \langle m_f \rangle$ .
- (c) If  $\langle g \rangle = \langle f \rangle$ , then  $m_f$  is a quasi right part of  $g$ .

The class of all isomorphisms in  $\mathcal{X}$  is denoted by  $Iso(\mathcal{X})$ .

**Proposition 1.3.** Suppose  $\mathcal{M}$  is closed under composition with isomorphisms, i.e.,  $m \in \mathcal{M}$ ,  $\alpha \in Iso(\mathcal{X})$ , and  $\alpha m$  defined, yields that  $\alpha m \in \mathcal{M}$ . If  $f$  is a morphism in  $\mathcal{X}$  and  $m_f$  is a quasi right part of  $f$ , then  $\alpha m_f$  is a quasi right part of  $\alpha f$ .

*Proof.* Obvious.  $\square$

**Notation 1.** The  $\mathcal{M}$ -part of a quasi right  $\mathcal{M}$ -factorization of the composite  $fm$ , where  $M \xrightarrow{m} X \xrightarrow{f} Y$ , is denoted by  $f(m) : f(M) \longrightarrow Y$ .

**Remark 1.4.** Suppose that  $\mathcal{X}$  has quasi right  $\mathcal{M}$ -factorizations.

- (a) For each  $M \xrightarrow{m} X \in \mathcal{M}$  we have  $\langle m(1_M) \rangle = \langle m \rangle$ .
- (b) For each  $X \xrightarrow{f} Y$ ,  $T \xrightarrow{g} Y$  and  $Y \xrightarrow{h} Z$  in  $\mathcal{X}$  if  $\langle f \rangle \subseteq \langle g \rangle$ , then  $\langle h(f) \rangle \subseteq \langle h(g) \rangle$ .

**Proposition 1.5.** Suppose that  $\mathcal{M}$  is closed under composition with isomorphisms. For each morphism  $X \xrightarrow{f} Y$  and isomorphism  $Y \xrightarrow{\alpha} Z$  in  $\mathcal{X}$  we have  $\langle \alpha(f) \rangle = \langle \alpha(f(1_X)) \rangle$ .

*Proof.* Obvious.  $\square$

**Definition 1.6.** (a) A morphism  $f$  is called *quasi monomorphism*, whenever for each morphism  $a, b \in \mathcal{X}$  if  $fa = fb$ , then  $\langle a \rangle = \langle b \rangle$ .

(b) A morphism  $f$  is called *quasi epimorphism*, whenever for each morphism  $a, b \in \mathcal{X}$  if  $af = bf$ , then  $\langle a \rangle = \langle b \rangle$ .

**Notation 2.** The classes of all quasi monomorphisms and quasi epimorphisms are denoted by  $QM(\mathcal{X})$  and  $QE(\mathcal{X})$ , respectively.

**Definition 1.7.** [10] Let  $\mathcal{E}$  be a class of morphisms in  $\mathcal{X}$ . We say that  $\mathcal{X}$  has *quasi left  $\mathcal{E}$ -factorizations* or  $\mathcal{E}$  is a *quasi left factorization structure* in  $\mathcal{X}$ , whenever for all morphisms  $f : Y \longrightarrow X$  in  $\mathcal{X}$  there exists  $Y \xrightarrow{e_f} M \in Y/\mathcal{E}$  such that:

- (a)  $f = ge_f$  for some  $g$ ;
- (b) if there exists  $e \in Y/\mathcal{E}$  such that  $f = g'e$  for some  $g'$ , then  $e_f = he$  for some  $h$ .

$e_f$  is called a quasi left part of  $f$ .

With  $\rangle e \langle$  denoting the cosieve generated by  $e$ , (a) is equivalent to:

(a')  $\rangle f \langle \subseteq \rangle e_f \langle$ ;

and (b) is equivalent to:

(b') if there exists  $e \in X/\mathcal{E}$  such that  $\rangle f \langle \subseteq \rangle e \langle$ , then  $\rangle e_f \langle \subseteq \rangle e \langle$ .

Note that the left  $\mathcal{E}$ -factorizations in [6] are quasi left  $\mathcal{E}$ -factorizations.

**Lemma 1.8.** [10] Suppose  $\mathcal{E}$  is a quasi left factorization structure in  $\mathcal{X}$ . Let  $f$  be a morphism in  $\mathcal{X}$  and  $e_f$  be a quasi left part of  $f$ .

- (a) If  $f \in \mathcal{E}$ , then  $\rangle e_f \langle = \rangle f \langle$ .
- (b)  $e$  is a quasi left part of  $f$  if and only if  $e \in \mathcal{E}$  and  $\rangle e \langle = \rangle e_f \langle$ .
- (c) If  $\rangle g \langle = \rangle f \langle$ , then  $e_f$  is a quasi left part of  $g$ .

## 2. GENERAL CLOSURE OPERATORS

Let  $\mathcal{M}$  be a class of morphisms in a category  $\mathcal{X}$ . Define a preorder on  $\mathcal{M}/X$  by  $f \leq g$  if  $\langle f \rangle \subseteq \langle g \rangle$  and define the equivalence relation on  $\mathcal{M}/X$  by  $f \sim g$  iff  $f \leq g$  and  $g \leq f$ . Assuming  $\mathcal{X}$  has quasi right  $\mathcal{M}$ -factorizations, we have:

**Definition 2.1.** [6], [9], A *general closure operator*  $\mathbf{C}$  on  $\mathcal{X}$  with respect to  $\mathcal{M}$  is given by  $\mathbf{C} = (\mathbf{c}_X)_{X \in \mathcal{X}}$ , where  $\mathbf{c}_X : \mathcal{M}/X \longrightarrow \mathcal{M}/X$  is a map satisfying:

- (a) the extension property: for all  $X \in \mathcal{X}$  and for all  $m \in \mathcal{M}/X$ ,  $m \leq \mathbf{c}_X(m)$ ;
- (b) the monotonicity property: for all  $X \in \mathcal{X}$  and for all  $m, m'$  in  $\mathcal{M}/X$  whenever  $m \leq m'$  in  $\mathcal{M}/X$ , then  $\mathbf{c}_X(m) \leq \mathbf{c}_X(m')$ ;
- (c) the continuity property: for all morphisms  $f : X \longrightarrow Y$  in  $\mathcal{X}$  and for all  $m \in \mathcal{M}/X$ ,  $f(\mathbf{c}_X(m)) \leq \mathbf{c}_Y(f(m))$ .

**Remark 2.2.** In the presence of (2), if  $\mathcal{M}$  has  $\mathcal{X}$ -pullbacks (i.e.  $m \in \mathcal{M}$  and  $f \in \mathcal{X}$  implies the pullback,  $f^{-1}(m)$ , of  $m$  along  $f$  is in  $\mathcal{M}$ ), then the continuity condition can equivalently be expressed as

- (c') for all morphisms  $f : X \longrightarrow Y$  in  $\mathcal{X}$  and for all  $m \in \mathcal{M}/Y$ ,

$$\mathbf{c}_X(f^{-1}(m)) \leq f^{-1}(\mathbf{c}_Y(m)).$$

**Example 2.3.** Consider the category  ${}_R M_S$  of  $(R, S)$ -bimodules, where  $R$  and  $S$  are commutative rings and suppose that there exists a ring homomorphism  $\sigma : R \longrightarrow S$  such that  $\sigma(1_R) = 1_S$ . Thus  $S$  is a  $R$ -module by  $r \cdot s = \sigma(r)s$  and hence  $S \in {}_R M_S$ . Suppose that  $\mathcal{C}$  is a full subcategory of  ${}_R M_S$  whose objects are  $(R, S)$ -bimodules  $M$  such that for each  $r \in R$ ,  $s \in S$  and  $m \in M$  we have  $s(rm) = (s \cdot r)m$ . Let  $\mathcal{M}$  be the class of retractions in  $\mathcal{C}$ . Then  $\mathcal{M}$  is a quasi right factorization structure in  $\mathcal{C}$ , [13]. A morphism  $f : X \longrightarrow Y$  in  $\mathcal{C}$  can be factored as follows:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \langle 0, f \rangle & \nearrow \pi_2 \\ & X \oplus Y & \end{array}$$

For each morphism  $\varphi : M \longrightarrow X$  in  $\mathcal{M}$ , define its closure to be the map  $\bar{\varphi}$ , which is the unique map making the following diagram commute,

$$\begin{array}{ccc}
S \times M & \xrightarrow{\otimes_R} & S \otimes_R M \\
\searrow \psi & \swarrow \varphi & \\
& X &
\end{array}$$

where the map  $\psi : S \times M \longrightarrow X$  takes  $(s, m)$  to  $s\varphi(m)$ .

**Example 2.4.** Let  $\mathcal{C}$  be the category of torsion free modules, [7], and  $\mathcal{M}$  be the class of retractions. Then  $\mathcal{M}$  is a quasi right factorization structure in  $\mathcal{C}$ , [13]. A morphism  $f : X \longrightarrow Y$  in  $\mathcal{C}$  can be factored as follows:

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\searrow \langle 0, f \rangle & \swarrow \pi_2 & \\
& X \oplus Y &
\end{array}$$

Suppose that  $m : M \longrightarrow X$  is a morphism in  $\mathcal{M}$ . There is a torsion free precover  $\varphi : T \longrightarrow X$ . Since  $M$  is torsion free, there is a map  $\psi : M \longrightarrow T$  making the following diagram commute:

$$\begin{array}{ccc}
M & \xrightarrow{\psi} & T \\
\searrow m & \swarrow \varphi & \\
& X &
\end{array}$$

Now define the closure of  $m$  to be the map  $\varphi$ .

**Example 2.5.** Let  $\mathcal{C}$  be an abelian category with enough injectives, [8]. The collection  $\mathcal{M}$  of all epimorphisms whose kernels are injective is a quasi right factorization structure. A morphism  $f : X \longrightarrow Y$  can be factored as  $\pi_2 \langle i, f \rangle$ , where  $i : X \longrightarrow E$  is the monomorphism from  $X$  to an injective object  $E$  and  $\pi_2 : E \times Y \longrightarrow Y$  is the projection to the second factor.

Now for each morphism  $m : M \longrightarrow X$  define its closure to be the map  $m\pi_2 : K \oplus M \twoheadrightarrow X$  where  $K = \text{Ker}(m)$ .

**Example 2.6.** Let  $\mathcal{C}$  be a closed model category, [14]. The collection  $\mathcal{M}$  of fibrations is a quasi right factorization structure. A morphism  $f : X \longrightarrow Y$  can be factored as  $pi$ , where  $p$  is a fibration and  $i$  is a weak equivalence.

For each object  $X \in \mathcal{C}$  we have a trivial fibration  $p_x : Q(X) \longrightarrow X$  with  $Q(X)$  cofibrant. Now for each morphism  $m : M \longrightarrow X$  in  $\mathcal{M}$  define its closure to be the map  $mp_m : Q(M) \longrightarrow X$ .

**Example 2.7.** As a special case of Example 2.6, in the category **Top**, of topological spaces and continuous maps, the collection  $\mathcal{M}$  of serre fibrations is a quasi right factorization structure. A morphism  $f : X \longrightarrow Y$  can be factored as  $pi$ , where  $p$  is a serre fibration and  $i$  is a homotopy equivalence.

Now the closure of a morphism  $m : M \longrightarrow X$  in  $\mathcal{M}$  is as in Example 2.6.

**Example 2.8.** Let  $\mathcal{C}$  be a *model category*. For the *category of fibrant objects*,  $\mathcal{C}_f$ , the collection  $\mathcal{M}$  of fibrations is a quasi right factorization structure. A morphism  $f : X \longrightarrow Y$  can be factored as  $pi$ , where  $p$  is a fibration and  $i$  is a trivial cofibration.

Define the closure of  $m : M \longrightarrow X$  in  $\mathcal{M}$  to be the projection to the first factor,  $\pi_1 : X \times M \longrightarrow X$ .

**Example 2.9.** As a special case of Example 2.8, in the category  $\mathbf{Top}$ , in which all the objects are fibrant, the collection  $\mathcal{M}$  of *serre fibrations* is a quasi right factorization structure. A morphism  $f : X \longrightarrow Y$  can be factored as  $pi$ , where  $p$  is a *serre fibration* and  $i$  is a trivial cofibration.

Define the closure of  $m : M \longrightarrow X$  in  $\mathcal{M}$  the projection to the first factor,  $\pi_1 : X \times M \longrightarrow X$ .

**Example 2.10.** In the *cofibrant category*  $(\mathbf{Top}, \text{cofibrations, homotopy equivalences})$ , the collection  $\mathcal{M}$  of homotopy equivalences is a quasi right factorization structure. A morphism  $f : X \longrightarrow Y$  can be factored as follows,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow i_f \quad \swarrow r_f & \\ & Z_f & \end{array}$$

where  $r_f$  is a homotopy equivalence,  $i_f$  is a cofibration and  $Z_f$  is the mapping cylinder of  $f$ , [11].

For each morphism  $m : M \longrightarrow X$  in  $\mathcal{M}$  define its closure to be the map,  $r_m : Z_m \longrightarrow X$ .

**Example 2.11.** In the *fibrant category*  $(\mathbf{Top}, \text{fibrations, homotopy equivalences})$ , the collection  $\mathcal{M}$  of homotopy equivalences is a quasi right factorization structure. A morphism  $f : X \longrightarrow Y$  can be factored as follows,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow q_f \quad \swarrow k_f & \\ & P_f & \end{array}$$

where  $q_f$  is a homotopy equivalence,  $k_f$  is a fibration and  $P_f$  is the mapping path space of  $f$ , [11].

For each morphism  $m : M \longrightarrow X$  in  $\mathcal{M}$  define its closure to be the map,  $k_m : P_m \longrightarrow X$ .

**Example 2.12.** [13]. In the *Kleisli category*  $\mathbf{Set}_{\mathbb{P}}$ , where  $P$  is the power set monad  $\mathbb{P} = (P, \eta, \mu)$ , for each morphism  $\hat{f} : X \longrightarrow Y$  in  $\mathbf{Set}_{\mathbb{P}}$ , let

$f : X \longrightarrow P(Y)$  be its associated morphism in  $Set$  and

$$X \xrightarrow{f} P(Y) = X \xrightarrow{f'} I_f \xrightarrow{m_f} P(Y)$$

be the  $(Epi, Mono)$  factorization of  $f$ . The class  $\mathcal{M} = \{\widehat{m}_f : \hat{f} \in Set_{\mathbb{P}}\}$  is a quasi right factorization structure. Each  $f : M \longrightarrow N$  can be factored as  $m(sf)$ , where  $s$  is any section of  $m$ .

For each morphism  $\widehat{m}_f : I_f \longrightarrow Y$  in  $\mathcal{M}$  define its closure to be the map  $\widehat{\mu_Y P(m_f)} : P(I_f) \longrightarrow Y$ .

**Example 2.13.** In the category **Top**, the class  $\mathcal{M} = \{h \oplus h : h \in Top\}$  is a quasi right factorization structure. A morphism  $f : X \longrightarrow Y$  can be factored as  $f = (f \oplus f)\nu_1$ , where  $\nu_1$  is the injection of the coproduct.

For each morphism  $m$  in  $\mathcal{M}$  define its closure to be the map  $m \oplus m$ .

**Example 2.14.** [13]. In the full subcategory **ProjRMod** of the category **RMod**, of  $R$ -modules and  $R$ -module homomorphisms, consisting of all projective  $R$ -modules, the collection  $\mathcal{M}$  of all epimorphisms with free domain is a quasi right factorization structure.

For each morphism  $m : F \longrightarrow P$  in  $\mathcal{M}$  define its closure to be the map

$$\langle m, m \rangle : F \oplus F \longrightarrow P$$

Now on instead of saying  $\mathbf{C}$  is a closure operator on the category  $\mathcal{X}$  with respect to  $\mathcal{M}$  we can say  $\mathbf{C}$  is a closure operator. Also we assume that all identities are in  $\mathcal{M}$  and  $\mathcal{X}$  has quasi right  $\mathcal{M}$ -factorizations.

**Definition 2.15.** Suppose that  $\mathbf{C}$  is a closure operator and  $m \in \mathcal{M}/X$ . We say  $m$  is

- (a) (see [13]) *quasi  $\mathbf{C}$ -closed* in  $X$ , if  $c_X(m) \sim m$ ;
- (b) *quasi  $\mathbf{C}$ -dense* in  $X$ , if  $c_X(m) \sim 1_X$ .

A morphism  $f$  in  $\mathcal{X}$  is called *quasi  $\mathbf{C}$ -dense*, whenever  $f(1_X)$  is quasi  $\mathbf{C}$ -dense in  $\mathcal{X}$ . We denote by  $\mathcal{E}^{QC}$ , the class of all quasi  $\mathbf{C}$ -dense morphisms in  $\mathcal{X}$ .

**Example 2.16.** (i) In the examples 2.10, 2.11, 2.12, members of  $\mathcal{M}$  are all quasi  $\mathbf{C}$ -closed.

(ii) In the examples 2.4, 2.5, 2.8, 2.10, 2.11, members of  $\mathcal{M}$  are all quasi  $\mathbf{C}$ -dense.

**Proposition 2.17.** (a) Suppose that  $\mathcal{M}$  is closed under composition with isomorphisms. For each morphism  $X \xrightarrow{f} Y \in \mathcal{E}^{QC}$  and isomorphism  $Y \xrightarrow{\alpha} Z$  in  $\mathcal{X}$  we have  $\alpha f \in \mathcal{E}^{QC}$ .

- (b) If  $f \leq g$  and  $f \in \mathcal{E}^{QC}$ , then  $g \in \mathcal{E}^{QC}$

*Proof.* (a) By Propositions 1.3 and 1.5 and the continuity property we have  $\alpha(\mathbf{c}_Y(f(1_X))) \leq \mathbf{c}_Z((\alpha f)(1_X))$ . Since  $\mathbf{c}_Y(f(1_X)) \sim 1_Y$  and  $\alpha(1_Y) \sim 1_Z$ , we have  $\alpha f \in \mathcal{E}^{QC}$ .

(b) Obvious.  $\square$

**Remark 2.18.** (a) For each  $m, n \in \mathcal{M}$  if  $m \sim n$  and  $m$  is  $\mathbf{C}$ -dense, then  $n$  is  $\mathbf{C}$ -dense.

(b) If  $\mathcal{M}$  is a class of monomorphisms, then  $m$  is quasi  $\mathbf{C}$ -closed (dense) if and only if  $m$  is  $\mathbf{C}$ -closed (dense).

### 3. QUASI IDEMPOTENT AND QUASI WEAKLY HEREDITARY CLOSURE OPERATORS

In this section we define quasi idempotent and quasi weakly hereditary closure operators and show which of the examples in the previous section has these properties. Finally we prove under what conditions on the closure operator we have a quasi right(left) factorization structure.

**Definition 3.1.** Suppose that  $\mathbf{C}$  is a closure operator.  $\mathbf{C}$  is called:

(a) (see [13]) *quasi idempotent*, if  $\mathbf{c}_X(\mathbf{c}_X(m)) \sim \mathbf{c}_X(m)$ , for each  $X \in \mathcal{X}$  and  $m \in \mathcal{M}/X$ .

(b) *quasi weakly hereditary*, if  $\mathbf{c}_Y(j_m(1_M)) \sim 1_Y$ , for each  $X \in \mathcal{X}$  and  $m \in \mathcal{M}/X$ , where

$$\begin{array}{ccc} M & \xrightarrow{j_m} & \mathbf{c}_X(M) = Y \\ & \searrow m \quad \swarrow \mathbf{c}_X(m) & \\ & X & \end{array} \quad \begin{array}{c} \text{///} \end{array}$$

**Example 3.2.** (i) In the examples 2.4, 2.5, 2.8, 2.10, 2.11, 2.12, the closure operator is quasi idempotent.

(ii) In the examples 2.4, 2.10, 2.11, 2.12, the closure operator is quasi weakly hereditary.

Let  $\mathcal{M}^{QC}$  be the class of quasi  $\mathbf{C}$ -closed members of  $\mathcal{M}$ . With  $(\mathcal{E}, \mathcal{M})$ -factorization structure as defined in [2], we have:

**Theorem 3.3.** [13] Suppose that  $\mathcal{X}$  has  $(\mathcal{E}, \mathcal{M})$ -factorization structure and  $\mathbf{C}$  is a quasi idempotent closure operator. Then  $\mathcal{M}^{QC}$  is a quasi right factorization structure for  $\mathcal{X}$ .

**Theorem 3.4.** For a quasi idempotent closure operator  $\mathbf{C}$ ,  $\mathcal{X}$  has quasi right  $\mathcal{M}^{QC}$ -factorization.

*Proof.* Every morphism  $X \xrightarrow{f} Y$  has a quasi right  $\mathcal{M}$ -factorization,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow e \quad \swarrow m & \\ & M & \end{array} \quad \begin{array}{c} \text{q.r.f} \end{array}$$



Since  $m \leq \mathbf{c}_Y(m)$ , we have  $M \xrightarrow{m} Y = M \xrightarrow{j_m} \mathbf{C}_Y(M) \xrightarrow{\mathbf{c}_Y(m)} Y$  and  $\mathbf{c}_Y(m) \in \mathcal{M}^{QC}$ . Consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ j_m e \downarrow & \quad \quad \quad & \downarrow n \\ \mathbf{C}_Y(M) & \xrightarrow{\mathbf{c}_Y(m)} & Y \end{array}$$

where  $n \in \mathcal{M}^{QC}$ . So  $f = nu$  and hence  $m \leq n$ . Thus  $\mathbf{c}_Y(m) \leq \mathbf{c}_Y(n) \sim n$ .

Therefore  $\mathbf{c}_Y(m) \leq n$  and  $X \xrightarrow{f} Y = X \xrightarrow{j_m e} \mathbf{C}_Y(M) \xrightarrow{\mathbf{c}_Y(m)} Y$  is a quasi right  $\mathcal{M}$ -factorization of  $f$ .  $\square$

**Remark 3.5.** For a closure operator  $\mathbf{C}$  we have  $\mathcal{M}^{QC} \cap \mathcal{E}^{QC} = \{f \in \text{Mor}(\mathcal{X}) \mid f \sim 1\}$ . If  $\mathcal{M} \subseteq QM(\mathcal{X})$ , then  $\mathcal{M}^{QC} \cap \mathcal{E}^{QC} = \text{Iso}(\mathcal{X})$ .

**Proposition 3.6.** Suppose that  $\mathcal{M} \subseteq QM(\mathcal{X})$  is closed under composition. If

$$X \xrightarrow{f} Y = X \xrightarrow{e} M \xrightarrow{m} Y$$

is a quasi right  $\mathcal{M}$ -factorization of  $f$ , then  $e \in \mathcal{E}^{QC}$ .

*Proof.* Let  $X \xrightarrow{e=e1_X} M = X \xrightarrow{e_1} e(X) \xrightarrow{e(1_X)} M$  be a right  $\mathcal{M}$ -factorization of  $e1_X$ . So we have the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{e_1} & e(X) \\ e \downarrow & \quad \quad \quad & \downarrow me(1_X) \\ M & \xrightarrow{m} & Y \end{array}$$

$w$  (dotted arrow from  $X$  to  $e(X)$ ),  $///$  (dotted arrow from  $M$  to  $e(X)$ )

Therefore  $m = me(1_X)w$  and hence  $\langle e(1_X)w \rangle = \langle 1_M \rangle$ . Thus  $1_M \leq e(1_X)$  and so  $e \in \mathcal{E}^{QC}$ .  $\square$

**Proposition 3.7.** Suppose that  $\mathcal{M} \subseteq QM(\mathcal{X})$ . If  $X \xrightarrow{e} M$  is quasi  $\mathbf{C}$ -dense, then  $\mathbf{c}_M(e(1_X))$  is an isomorphism.

*Proof.* Let  $X \xrightarrow{e=e1_X} M = X \xrightarrow{e_1} e(X) \xrightarrow{e(1_X)} M$  be a right  $\mathcal{M}$ -factorization of  $e1_X$ . Since  $e \in \mathcal{E}^{QC}$  and  $\mathbf{c}_M(e(1_X)) \sim 1_M$ , there exist morphisms  $f$  and  $g$  such that the following commutative triangle commutes

$$\begin{array}{ccc} \mathbf{c}_M(e(X)) & \xrightarrow{f} & M \\ & \quad \quad \quad & \uparrow g \\ \mathbf{c}_M(e(1_X)) & \xrightarrow{\quad} & M \end{array}$$

$///$  (dotted arrow from  $\mathbf{c}_M(e(X))$  to  $M$ ),  $1_M$  (arrow from  $M$  to  $M$ )

Thus  $\mathbf{c}_M(e(1_X)) = f$  and  $\mathbf{c}_M(e(1_X))g = 1_M$ ; hence  $fg = 1_M$  and  $fgf = f$ . Since  $f \in \mathcal{M}$ , we have  $\langle gf \rangle = \langle 1_{\mathbf{c}_M(e(X))} \rangle$  and there exists a morphism  $h$  such that  $ghf = 1_{\mathbf{c}_M(e(X))}$ . Therefore  $f$  is an isomorphism.  $\square$

**Notation 3.** Let  $\mathcal{E}$  and  $\mathcal{M}$  be two classes of morphisms in the category  $\mathcal{X}$ , with  $e, e' \in \mathcal{E}$  and  $m \in \mathcal{M}$ . We write,  $e' \overset{e}{\bar{\vee}} m$ , whenever in the unbroken commutative diagram:

$$\begin{array}{ccc} & \xrightarrow{e} & \\ e' \downarrow & \swarrow & \downarrow m \\ & \xrightarrow{d} & \end{array}$$

there exists a morphism  $d$  such that  $e = de'$ .

**Remark 3.8.**  $e' \overset{e}{\bar{\vee}} m$  is equivalent to; if  $\rangle me \langle \subseteq \rangle e' \langle$ , then  $\rangle e \langle \subseteq \rangle e' \langle$ .

Now we can define the classes  $\overset{\mathcal{E}}{\bar{\vee}} \mathcal{M}$  as follows:

$$\overset{\mathcal{E}}{\bar{\vee}} \mathcal{M} := \{e' \in \mathcal{E} \mid e' \overset{e}{\bar{\vee}} m, \forall e \in \mathcal{E} \text{ and } \forall m \in \mathcal{M}\}.$$

For a closure operator  $\mathbf{C}$  consider the following property:

(QCD) Composites of quasi  $\mathbf{C}$ -dense morphisms are quasi  $\mathbf{C}$ -dense, i.e., if  $M \xrightarrow{m} N$  and  $N \xrightarrow{n} X$  in  $\mathcal{M}$  are quasi  $\mathbf{C}$ -dense, then  $nm$  is quasi  $\mathbf{C}$ -dense.

**Theorem 3.9.** Suppose that  $\mathbf{C}$  is a quasi weakly hereditary closure operator, (QCD) holds for every  $X \in \mathcal{X}$ ,  $\mathcal{E}^{QC} \subseteq QE(\mathcal{X})$  and  $\mathcal{E}^{QC} \subseteq \overset{\mathcal{E}^{QC}}{\bar{\vee}} \mathcal{M}$ . Then  $\mathcal{X}$  has quasi left  $\mathcal{E}^{QC}$ -factorization structures.

*Proof.* Suppose that  $X \xrightarrow{f} Y = X \xrightarrow{e} M \xrightarrow{m} Y$  is a quasi right  $\mathcal{M}$ -factorization of  $f$  and  $m = \mathbf{c}_Y(m)j_m$ . Put  $d := j_me$ . First we show that  $d \in \mathcal{E}^{QC}$ . Let

$$X \xrightarrow{d} \mathbf{c}_Y(M) = X \xrightarrow{e_1} d(X) \xrightarrow{d(1_X)} \mathbf{c}_Y(M)$$

and

$$M \xrightarrow{j_m} \mathbf{c}_Y(M) = M \xrightarrow{e'_1} j_m(M) \xrightarrow{j_m(1_M)} \mathbf{c}_Y(M)$$

be the right  $\mathcal{M}$ -factorizations of  $d$  and  $j_m$ , respectively. By Proposition 3.6 we have  $\{e, e_1, e'_1\} \subseteq \mathcal{E}^{QC}$  and since  $d(1_X)e_1 = j_m(1_M)e'_1e$ , there exists

$$w : j_m(M) \longrightarrow d(X)$$

such that  $we'_1e = e_1$ . Thus  $d(1_X)we'_1e = j_m(1_M)e'_1e$  and hence  $\langle d(1_X)w \rangle = \langle j_m(1_M) \rangle$ . Therefore  $j_m(1_M) \leq d(1_X)$  and since  $\mathbf{C}$  is quasi weakly hereditary, we have  $d \in \mathcal{E}^{QC}$ . Given the commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{e'} & Z \\ j_me \downarrow & \swarrow & \downarrow u \\ \mathbf{c}_Y(M) & \xrightarrow{\mathbf{c}_Y(m)} & Y \end{array}$$

where  $e' \in \mathcal{E}^{QC}$ . Consider the quasi right  $\mathcal{M}$ -factorization of  $u$  as follows.

$$\begin{array}{ccc} Z & \xrightarrow{u} & Y \\ & \searrow e'' \quad \swarrow m_u & \\ & N & \end{array} \quad \text{q.r.f.}$$

So we have the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{j_me} & \mathbf{c}_Y(M) \\ e''e' \downarrow & \swarrow w' & \downarrow \mathbf{c}_Y(m) \\ N & \xrightarrow{m_u} & Y. \end{array}$$

Thus  $w'e''e' = j_me$  and hence  $\langle j_me \rangle \subseteq \langle e' \rangle$ . Therefore  $f = \mathbf{c}_Y(m)(j_me)$  is a quasi left  $\mathcal{E}^{QC}$ -factorization of  $f$  and the proof is complete.  $\square$

#### 4. QUASI FACTORIZATION STRUCTURES

In this section the notations  $\mathcal{H}^\Delta$  and  ${}^\nabla\mathcal{H}$  are introduced and after studying some of their properties the notion of quasi factorization structure in a category  $\mathcal{X}$  is given. We will see that weak factorization structures as defined in [1] are quasi factorization structures, but the converse is not true as we will show by some examples. Finally we state the relation between a quasi factorization structure and a quasi idempotent and quasi weakly hereditary closure operator.

**Notation 4.** Let  $\mathcal{E}$  and  $\mathcal{M}$  be classes of morphisms in  $\mathcal{X}$ .

- (a) Given  $E \xrightarrow{e} X \in \mathcal{E}$  and  $M \xrightarrow{m} X \in \mathcal{M}$ ,  
 $e \nabla m$  means that in every commutative triangle:

$$\begin{array}{ccc} E & \xrightarrow{u} & M \\ & \searrow e \quad \swarrow m & \\ & X & \end{array} \quad ///$$

there exists  $w : X \longrightarrow M$  such that  $mw \sim 1_X$ .

- (b) Given  $M \xrightarrow{e} E \in \mathcal{E}$  and  $M \xrightarrow{m} X \in \mathcal{M}$ ,  
 $e \Delta m$  means that in every commutative triangle:

$$\begin{array}{ccc} & M & \\ e \swarrow & & \searrow m \\ E & \xrightarrow{v} & X \end{array} \quad ///$$

there exists  $w : E \longrightarrow M$  such that  $\langle mw \rangle = \langle v \rangle$ .

Let  $\mathcal{H}$  be a class of morphisms. We denote by  $\mathcal{H}^\Delta$  the class of all morphisms  $m$  with

$$h \Delta m \quad \text{for all } h \in \mathcal{H}$$

and dually, by  $\nabla \mathcal{H}$  the class of all morphisms  $e$  with

$$e \nabla h \quad \text{for all } h \in \mathcal{H}.$$

**Proposition 4.1.** If  $\mathcal{H}_1 \subseteq \mathcal{H}_2$ , then  $\nabla \mathcal{H}_2 \subseteq \nabla \mathcal{H}_1$  and  $\mathcal{H}_2^\Delta \subseteq \mathcal{H}_1^\Delta$ .

*Proof.* Obvious.  $\square$

**Proposition 4.2.** Suppose that  $\mathcal{M} \subseteq QM(\mathcal{X})$  is closed under composition. If  $\mathbf{C}$  is a quasi idempotent closure operator, then  $\mathcal{E}^{QC} \subseteq \nabla(\mathcal{M}^{QC})$ .

*Proof.* Let  $e \in \mathcal{E}^{QC}$  and  $m \in \mathcal{M}^{QC}$  be given such that the following triangle commutes.

$$\begin{array}{ccc} E & \xrightarrow{u} & M \\ & \searrow e \quad \swarrow m & \\ & X & \end{array} \quad \begin{array}{c} \text{///} \end{array}$$

Consider the quasi right  $\mathcal{M}$ -factorization of  $e$  as follows.

$$E \xrightarrow{e} X = E \xrightarrow{e_1} e(E) \xrightarrow{e(1_E)} X$$

We have  $e(1_E) = \mathbf{c}_X(e(1_E))j$ . By Proposition 3.7,  $\mathbf{c}_X(e(1_E))$  is an isomorphism and Theorem 3.4 implies that the following factorization:

$$\begin{array}{ccc} E & \xrightarrow{e} & X \\ & \searrow j e_1 \quad \swarrow \mathbf{c}_X(e(1_E)) & \\ & \mathbf{c}_X(e(E)) & \end{array} \quad \begin{array}{c} \text{q.r.f} \end{array}$$

is a quasi right  $\mathcal{M}$ -factorization of  $e$ . So we have the following commutative diagram.

$$\begin{array}{ccc} E & \xrightarrow{u} & M \\ j e_1 \downarrow & \swarrow w_1 \quad \searrow m & \\ \mathbf{c}_X(e(E)) & \xrightarrow{\mathbf{c}_X(e(1_E))} & X \end{array} \quad \begin{array}{c} \text{///} \end{array}$$

Put  $w := \stackrel{\text{def}}{=} w_1 \mathbf{c}_X(e(1_E))^{-1}$ , so  $mw = 1_X$ . Therefore  $\mathcal{E}^{QC} \subseteq \nabla(\mathcal{M}^{QC})$ .  $\square$

**Proposition 4.3.** Suppose that  $\mathcal{M} \subseteq QM(\mathcal{X})$ . If  $\mathbf{C}$  is a quasi weakly hereditary closure operator, (QCD) holds for every  $X \in \mathcal{X}$ ,  $\mathcal{E}^{QC} \subseteq QE(\mathcal{X})$  and  $\mathcal{E}^{QC} \subseteq \nabla \mathcal{M}$ , then  $\mathcal{M}^{QC} \subseteq (\mathcal{E}^{QC})^\Delta$ .

*Proof.* Let  $e \in \mathcal{E}^{QC}$  and  $m \in \mathcal{M}^{QC}$  be given such that the following triangle commutes.

$$\begin{array}{ccc} & M & \\ e \swarrow & & \searrow m \\ E & \xrightarrow{v} & X \end{array} \quad \begin{array}{c} \text{///} \end{array}$$

Since  $\mathbf{c}_X(m) \sim m$ , there exists  $g : \mathbf{c}_X(M) \longrightarrow M$  such that  $\mathbf{c}_X(m) = mg$  and hence  $mgj_m = m$ , where  $m = \mathbf{c}_X(m)j_m$ . Thus  $gj_m \sim 1_M$ . By Lemma 1.2 and Theorem 3.9 the following factorization:

$$\begin{array}{ccc} M & \xrightarrow{m} & X \\ & \searrow j_m \quad \swarrow \mathbf{c}_X(m) & \\ & \mathbf{c}_X(M) & \end{array} \quad \text{q.l.f.}$$

is a quasi left  $\mathcal{E}^{QC}$ -factorization of  $m$ . So we have the following diagram.

$$\begin{array}{ccc} M & \xrightarrow{e} & E \\ j_m \downarrow & \swarrow \text{dotted} & \downarrow v \\ \mathbf{c}_X(M) & \xrightarrow[\mathbf{c}_X(m)]{} & Y \end{array} \quad \begin{array}{c} e' \\ d' \end{array}$$

such that  $j_m = d'e$ . Put  $d \stackrel{\text{def}}{=} gd'$ . Thus  $md = mgd' = \mathbf{c}_X(m)d'$  and so  $mde = \mathbf{c}_X(m)d'e = ve$ . Since  $\mathcal{E}^{QC} \subseteq QE(\mathcal{X})$ , we have  $\langle md \rangle = \langle v \rangle$ .  $\square$

**Proposition 4.4.** For any two classes of morphisms  $\mathcal{E}$  and  $\mathcal{M}$ , we have  $\nabla \mathcal{M} \subseteq \mathcal{E}^{QC}$ .

*Proof.* Suppose that  $X \xrightarrow{e} M = X \xrightarrow{e_1} N \xrightarrow{e(1_X)} M$  is a quasi right  $\mathcal{M}$ -factorization of  $e$ . Since  $e \nabla e(1_X)$ , there exists  $w : M \longrightarrow N$  such that  $e(1_X)w \sim 1_M$ . Thus  $1_M \leq \mathbf{c}_M(e(1_X))$  and hence  $e \in \mathcal{E}^{QC}$ .  $\square$

In the following definition  $\mathcal{X}$  need not have quasi right  $\mathcal{M}$ -factorizations.

**Definition 4.5.** A *quasi factorization structure* in a category  $\mathcal{X}$  is a pair  $(\mathcal{E}, \mathcal{M})$  of classes of morphisms such that;

(a) every morphism  $f$  has a factorization as,

$$X \xrightarrow{f} Y = X \xrightarrow{e} M \xrightarrow{m} Y$$

where  $e \in \mathcal{E}$  and  $m \in \mathcal{M}$ ;

(b)  $\mathcal{E} = \nabla \mathcal{M}$  and  $\mathcal{M} = \mathcal{E}^\Delta$ .

**Example 4.6.** If  $(\mathcal{E}, \mathcal{M})$  is a weak factorization structure in a category  $\mathcal{X}$ , then it is a quasi factorization structure.

In particular, in the Examples 2.6 and 2.7 above, let  $\mathcal{E}$  be the class of weak equivalences. Then  $(\mathcal{E}, \mathcal{M})$  is a weak factorization structure and so is a quasi factorization structure.

In the following Examples  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure which is not a weak factorization structure.

**Example 4.7.** In the Example 2.10 above, let  $\mathcal{E}$  be the class of cofibrations. Then  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure.

To show  $(\mathcal{E}, \mathcal{M})$  is not a weak factorization structure, let  $\tau$  and  $\tau'$  be topologies on  $X$  with  $\tau' \subsetneq \tau$ . Consider the following commutative diagram:

$$\begin{array}{ccc} (X, \tau) & \xrightarrow{1_X} & (X, \tau) \\ 1_X \downarrow & \quad \quad \quad & \downarrow 1_X \\ (X, \tau') & \xrightarrow{1_X} & (X, \tau') \end{array} \quad \quad \quad \begin{array}{c} \text{///} \\ \text{///} \\ \text{///} \end{array}$$

The square has no diagonal, since otherwise if  $d : (X, \tau') \longrightarrow (X, \tau)$  is a diagonal, then  $d = 1_X$  and hence  $\tau \subseteq \tau'$  that implies  $\tau = \tau'$  which is a contradiction.

**Example 4.8.** In the Example 2.11 above, let  $\mathcal{E}$  be the class of fibrations. Then  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure. Since isomorphisms are also fibrations, similar to the Example 4.7, we can show that it is not a weak factorization structure.

**Example 4.9.** In the Example 2.12 above, let  $\mathcal{E} = \{\hat{e}_f : \hat{f} \in \text{Set}_{\mathbb{P}}\}$ , where  $e_f = \eta_{I_f} f'$ . Then  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure.

To show  $(\mathcal{E}, \mathcal{M})$  is not a weak factorization structure, let  $X = \{x, x', x''\}$  and consider the map  $f : X \longrightarrow P(X)$  taking all the points to  $\{x\}$ . Let

$$\begin{array}{ccc} X & \xrightarrow{\hat{f}} & X \\ & \searrow \hat{k} \quad \quad \nearrow \hat{m}_h & \\ & I_h & \end{array} \quad \quad \quad \begin{array}{c} \text{///} \\ \text{///} \\ \text{///} \end{array}$$

be an  $(\mathcal{E}, \mathcal{M})$  factorization of  $\hat{f}$ . As proved in [13] we can see the following commutative diagram has no diagonal.

$$\begin{array}{ccc} X & \xrightarrow{\hat{u}} & I_g \\ \hat{k} \downarrow & \quad \quad \quad & \downarrow \hat{m}_g \\ I_h & \xrightarrow{\hat{v}\hat{m}_h} & X \end{array} \quad \quad \quad \begin{array}{c} \text{///} \\ \text{///} \\ \text{///} \end{array}$$

**Example 4.10.** Let  $\mathcal{X}$  be a category with binary products in which projections are retractions. Let  $\mathcal{E} = \text{Sec}$  and  $\mathcal{M} = \text{Ret}$ . Where  $\text{Sec}$  and  $\text{Ret}$  denote the collection of all the sections and retractions, respectively. Then  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure. Each morphism  $f : X \longrightarrow Y$  can be factored as  $f = \pi_2 \langle 1, f \rangle$ , where  $\pi_2$  is the second projection.

To show  $(\text{Sec}, \text{Ret})$  is not generally a weak factorization structure, let  $\text{Top} - \{\emptyset\}$  be the full subcategory of  $\text{Top}$  consisting of the non-empty topological spaces and consider the following commutative diagram,

$$\begin{array}{ccc}
\{0\} & \xrightarrow{u} & \{0, 1, 2\} \\
s \downarrow & & \downarrow r \\
\{0, 1\} & \xrightarrow{\tau} & \{0, 1\}
\end{array}$$

where  $u$  sends 0 to 1, with codomain having  $\{1\}$  open;  $s$  is the inclusion with codomain having  $\{1\}$  open;  $r$  sends 0 to 0 and 1 and 2 to 1 with codomain having indiscrete topology; and  $\tau$  is the twist map. It is easy to see that  $s$  is a section and  $r$  is a retraction. The square has no diagonal, since otherwise if  $d$  is a diagonal, then  $ds = u$  and  $rd = \tau$ . It follows that  $d(0) = 1$  and  $d(1) = 0$ . Since  $d^{-1}(\{1\}) = \{0\}$ ,  $d$  is not continuous.

**Example 4.11.** Let  $\mathcal{X}$  be a category with coproducts and

$\mathcal{E} = \{ \nu_1 : A \longrightarrow A \amalg B : \nu_1 \text{ is the coproduct inclusion to the first factor} \}$

and  $\mathcal{M}$  be any collection of retractions. Then  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure.

To show  $(\mathcal{E}, \mathcal{M})$  is not generally a weak factorization structure, let  $\mathcal{X} = \text{Top} - \{\emptyset\}$ . Then the collection,

$\{ \nu_1 : A \longrightarrow A \amalg B : \nu_1 \text{ is the coproduct inclusion to the first factor} \}$

is the collection  $\text{Sec}$  of all the sections. Now the above example shows that it is not a weak factorization structure.

**Example 4.12.** Let  $\mathcal{X}$  be an abelian category. Define:

$$\mathcal{E} = \{ \langle 0, f \rangle : A \longrightarrow A \times B : f \in \mathcal{X} \}$$

and

$$\mathcal{M} = \{ \pi_2 : A \times B \longrightarrow B : \pi_2 \text{ is the second projection} \}$$

Then  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure. Each map  $f : X \longrightarrow Y$  can be factored as  $f = \pi_2 \langle 0, f \rangle$ .

To show  $(\mathcal{E}, \mathcal{M})$  is not generally a weak factorization structure, let  $A$  be a non-zero object. Then the commutative square:

$$\begin{array}{ccc}
A & \xrightarrow{\langle 1, 0 \rangle} & A \times 0 \\
\langle 0, 0 \rangle \downarrow & & \downarrow 0 \\
A \times 0 & \xrightarrow{0} & 0
\end{array}$$

has no diagonal, because if  $d = \langle d_1, d_2 \rangle$  is a diagonal, then  $d_1 \langle 0, 0 \rangle = 1$ , implying  $\langle 0, 0 \rangle$  is mono, which is a contradiction.

Saying  $\mathcal{E}$  has  $\mathcal{X}$ -pushouts if the pushout of each morphism in  $\mathcal{E}$  exists and is in  $\mathcal{E}$ , we have:

**Theorem 4.13.** Suppose that  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure in  $\mathcal{X}$ .

(a) If  $\mathcal{M}$  has  $\mathcal{X}$ -pullbacks, then  $\mathcal{X}$  has a quasi right  $\mathcal{M}$ -factorization structure.

(b) If  $\mathcal{M} \subseteq \text{Mon}(\mathcal{X})$  and  $\mathcal{E}$  has  $\mathcal{X}$ -pushouts, then  $\mathcal{X}$  has a quasi left  $\mathcal{E}$ -factorization structure.

*Proof.* Consider the quasi factorization of  $f$  as follows,

$$X \xrightarrow{f} Y = X \xrightarrow{e_f} M \xrightarrow{m_f} Y$$

where  $e_f \in \mathcal{E}$  and  $m_f \in \mathcal{M}$ . Suppose that the following unbroken square is commutative,

$$\begin{array}{ccc} X & \xrightarrow{u} & N \\ e_f \downarrow & \nearrow d & \downarrow n \\ M & \xrightarrow{m_f} & Y \end{array}$$

where  $n \in \mathcal{M}$ . So we have the following pullback diagram,

$$\begin{array}{ccccc} X & & & & N \\ & \searrow \exists! t & & \nearrow & \\ & & m_f^{-1}(N) & \xrightarrow{m'} & N \\ & \nearrow & \downarrow m_f^{-1}(n) & \text{p.b.} & \downarrow n \\ X & \xrightarrow{e} & M & \xrightarrow{m_f} & Y \end{array}$$

and the following triangle:

$$\begin{array}{ccc} X & \xrightarrow{t} & m_f^{-1}(N) \\ & \searrow e & \nearrow w \\ & & M \end{array}$$

Since  $m_f^{-1}(n) \in \mathcal{M}$  and  $\mathcal{E} = \nabla \mathcal{M}$ , there exists  $w : M \longrightarrow m_f^{-1}(N)$  such that  $m_f^{-1}(n)w \sim 1_M$ . Thus there exists a morphism  $\alpha : M \longrightarrow M$  such that  $m_f^{-1}(n)w\alpha = 1_M$ . Now define  $d \stackrel{\text{def}}{=} m'w\alpha$ . So we have  $nd = nm'w\alpha = m_fm_f^{-1}(n)w\alpha = m_f$ .

(b) Consider the quasi factorization of  $f$  as follows,

$$X \xrightarrow{f} Y = X \xrightarrow{e_f} M \xrightarrow{m_f} Y$$

where  $e_f \in \mathcal{E}$  and  $m_f \in \mathcal{M}$ . Suppose that the following unbroken square is commutative

$$\begin{array}{ccc} X & \xrightarrow{e} & E \\ e_f \downarrow & \nearrow d' & \downarrow v \\ M & \xrightarrow{m_f} & Y \end{array}$$



where  $e \in \mathcal{E}$ . So we have the following pushout diagram,

$$\begin{array}{ccc}
 X & \xrightarrow{e} & E \\
 \downarrow e_f & \text{p.o.} & \downarrow e' \\
 M & \xrightarrow{e''} & E' \\
 & \text{///} & \text{---} \exists! t' \text{---} \\
 & & Y
 \end{array}
 \quad \begin{array}{c}
 \curvearrowright v \\
 \curvearrowleft m_f
 \end{array}$$

and the following triangle:

$$\begin{array}{ccc}
 & M & \\
 \swarrow w' & & \searrow m_f \\
 E' & \xrightarrow{t'} & Y
 \end{array}
 \quad \begin{array}{c}
 \text{---} e'' \text{---}
 \end{array}$$

Since  $e'' \in \mathcal{E}$  and  $\mathcal{M} = \mathcal{E}^\Delta$ , there exists  $w' : E' \longrightarrow M$  such that  $\langle m_f w' \rangle = \langle t' \rangle$ . Thus there exists  $\beta : E' \longrightarrow E'$  such that  $m_f w' \beta = t'$ . Now define  $d' \stackrel{\text{def}}{=} w' \beta e'$ . So we have  $m_f w' \beta e' e = m_f w' \beta e' e_f = t' e' e_f = m_f e_f$ . Therefore  $d' e = e_f$ .  $\square$

Calling  $\mathcal{M}$ ,  $\sim$ -closed, if whenever  $m \in \mathcal{M}$  and  $f \sim m$ , then  $f \in \mathcal{M}$ , we have:

**Corollary 4.14.** Suppose  $\mathcal{M} \subseteq QM(\mathcal{X})$  is closed under composition and is  $\sim$ -closed, the closure operator  $\mathcal{C}$  is quasi weakly hereditary and quasi idempotent, and (QCD) holds for every  $X \in \mathcal{X}$ . If  $\mathcal{E}^{QC} \subseteq QE(\mathcal{X})$  and  $\mathcal{E}^{QC} \subseteq \bigvee_{\mathcal{E}^{QC}} \mathcal{M}$ , then  $(\mathcal{E}^{QC}, \mathcal{M}^{QC})$  is a quasi factorization structure in  $\mathcal{X}$ .

*Proof.* Consider the quasi right  $\mathcal{M}$ -factorization of  $f$  as follows:

$$X \xrightarrow{f} Y = X \xrightarrow{e_f} M \xrightarrow{m_f} Y$$

By Theorems 3.4 and 3.9, the factorization:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \searrow j_{m_f} e_f & & \nearrow c_Y(m_f) \\
 & c_Y(M_f) &
 \end{array}$$

is both quasi right  $\mathcal{M}^{QC}$ -factorization and quasi left  $\mathcal{E}^{QC}$ -factorization of  $f$ . By Propositions 4.2 and 4.3 it enough to show that  $\bigvee(\mathcal{M}^{QC}) \subseteq \mathcal{E}^{QC}$  and  $(\mathcal{E}^{QC})^\Delta \subseteq \mathcal{M}^{QC}$ . First we prove that  $\bigvee(\mathcal{M}^{QC}) \subseteq \mathcal{E}^{QC}$ . For this reason let

$a \in \nabla(\mathcal{M}^{QC})$  be given. So for each  $t \in \mathcal{M}^{QC}$  we have  $a \nabla t$ . Consider the quasi right  $\mathcal{M}$ -factorization of  $a$  as follows:

$$A \xrightarrow{a} T = A \xrightarrow{e'} K \xrightarrow{a(1_A)} T.$$

Thus we have the following commutative triangle,

$$\begin{array}{ccc} A & \xrightarrow{je'} & \mathbf{c}_T(K) \\ & \searrow a & \swarrow \mathbf{c}_T(a(1_A)) \\ & T & \end{array} \quad \begin{array}{c} \text{////} \end{array}$$

where  $a(1_A) = \mathbf{c}_T(a(1_A))j$ . Therefore, there exists  $w : T \longrightarrow \mathbf{c}_T(K)$  such that  $\mathbf{c}_T(a(1_A))w \sim 1_T$ . Hence  $1_T \leq \mathbf{c}_T(a(1_A))$ . Thus  $a \in \mathcal{E}^{QC}$ .

Now let  $b \in (\mathcal{E}^{QC})^\Delta$  be given. So for each  $s \in \mathcal{E}^{QC}$  we have  $s \Delta b$ . By Theorems 3.4 and 3.9 there exists a factorization:

$$B \xrightarrow{b} Q = B \xrightarrow{e_1} E_1 \xrightarrow{m_1} Q$$

such that  $e_1 \in \mathcal{E}^{QC}$  and  $m_1 \in \mathcal{M}^{QC}$ . Thus  $e_1 \Delta b$  and so there exists  $w_1 : E_1 \longrightarrow B$  such that  $\langle bw_1 \rangle = \langle m_1 \rangle$ . Thus  $m_1 \leq b$  and since  $b \leq m_1$ , we have  $b \sim m_1$ . Therefore  $b \in \mathcal{M}^{QC}$ .  $\square$

In [6], it is proved that if the category  $\mathcal{X}$  has  $(\mathcal{E}, \mathcal{M})$ -factorization structures and  $\mathbf{C}$  is a closure operator on  $\mathcal{X}$ , then  $\mathbf{C}$  is idempotent and weakly hereditary iff  $\mathcal{X}$  has  $(\mathcal{E}^C, \mathcal{M}^C)$ -factorizations. In the following we prove a similar result under weaker conditions.

**Theorem 4.15.** Suppose that  $\mathcal{M} \subseteq QM(\mathcal{X})$  is closed under composition, the closure operator  $\mathbf{C}$  is quasi weakly hereditary and quasi idempotent, and (QCD) holds for every  $X \in \mathcal{X}$ . If  $(\mathcal{E}, \mathcal{M})$  is a quasi factorization structure, then so is  $(\mathcal{E}^{QC}, \mathcal{M}^{QC})$ .

*Proof.* Consider the quasi factorization of  $f$  as follows,

$$X \xrightarrow{f} Y = X \xrightarrow{e} M \xrightarrow{m} Y$$

where  $e \in \mathcal{E}$  and  $m \in \mathcal{M}$ . Proposition 4.4 implies that  $e \in \mathcal{E}^{QC}$ . Also we have:  $X \xrightarrow{f} Y = X \xrightarrow{j_m e} \mathbf{c}_Y(M) \xrightarrow{\mathbf{c}_Y(m)} Y$ , where  $m = \mathbf{c}_Y(m)j_m$  and  $\mathbf{c}_Y(m) \in \mathcal{M}^{QC}$ . Put  $d \stackrel{\text{def}}{=} j_m e$ . Let

$$X \xrightarrow{d} \mathbf{c}_Y(M) = X \xrightarrow{e'} d(X) \xrightarrow{d(1_X)} \mathbf{c}_Y(M)$$

and

$$M \xrightarrow{j_m} \mathbf{c}_Y(M) = M \xrightarrow{e'_1} j_m(M) \xrightarrow{j_m(1_M)} \mathbf{c}_Y(M)$$

be the right  $\mathcal{M}$ -factorizations of  $d$  and  $j_m$ , respectively. Thus there exists a morphism  $w : d(X) \longrightarrow j_m(M)$  such that  $d(1_X) = j_m(1_M)w$ . Therefore

$\langle we' \rangle = \langle e'_1 e \rangle$  and hence there exists  $g : X \longrightarrow X$  such that  $we'g = e'_1 e$ . Let

$$\begin{array}{ccc} d(X) & \xrightarrow{w} & j_m(M) \\ & \searrow e'' \quad \nearrow w(1_{d(X)}) & \\ & M' & \end{array} \quad \text{q.r.f.}$$

be a quasi right  $\mathcal{M}$ -factorization of  $w$ . Since  $w(1_{d(X)}) \in \mathcal{M}$  and  $e'_1 e \nabla w(1_{d(X)})$ , there exists  $w' : j_m(M) \longrightarrow M'$  such that  $w(1_{d(X)})w' \sim 1_{j_m(M)}$ . Thus  $1_{j_m(M)} \leq w(1_{d(X)})$  and hence  $w \in \mathcal{E}^{QC}$ . Since  $d(1_X) = j_m(1_M)w$ , we have  $d \in \mathcal{E}^{QC}$ .

By Propositions 4.1 and 4.4 we have  $\mathcal{E}^{QC} \subseteq \nabla(\mathcal{M}^{QC})$  and  $\mathcal{M}^{QC} \subseteq (\mathcal{E}^{QC})^\Delta$ . As in the proof of the Corollary 4.14, it is easy to see that  $\nabla(\mathcal{M}^{QC}) \subseteq \mathcal{E}^{QC}$ . Now we show that  $(\mathcal{E}^{QC})^\Delta \subseteq \mathcal{M}^{QC}$ . Since  $\mathcal{E} = \nabla \mathcal{M}$ , by Proposition 4.4 we have  $\mathcal{E} \subseteq \mathcal{E}^{QC}$ . So Proposition 4.1 implies that  $(\mathcal{E}^{QC})^\Delta \subseteq \mathcal{E}^\Delta = \mathcal{M}$ . Let  $b \in (\mathcal{E}^{QC})^\Delta$  be given. Thus  $b \in \mathcal{M}$  and for each  $s \in \mathcal{E}^{QC}$  we have  $s\Delta b$ . In the following commutative triangle:

$$\begin{array}{ccc} B & \xrightarrow{j_b} & c_Q(B) \\ & \searrow b \quad \swarrow c_Q(b) & \\ & T & \end{array} \quad \text{///}$$

we have  $j_b \in \mathcal{E}^{QC}$ . Therefore  $j_b \Delta b$  and hence there exists  $w : c_Q(B) \longrightarrow B$  such that  $\langle bw \rangle = \langle c_Q(b) \rangle$ . Thus  $\langle c_Q(b) \rangle \subseteq \langle b \rangle$ , and so  $c_Q(b) \sim b$ . This implies that  $b \in \mathcal{M}^{QC}$ .  $\square$

**Definition 4.16.** (a)  $\mathcal{M}$  is called a *codomain* if  $m \in \mathcal{M}$  and  $\rangle m \langle \subseteq \rangle a \langle$ , yields  $a \in \mathcal{M}$ .

(b) A morphism  $f$  is called a *strongly quasi monomorphism*, whenever for all morphisms  $a, b \in \mathcal{X}$  if  $fa = fb$ , then  $\langle a \rangle = \langle b \rangle$  and  $\rangle a \langle = \rangle b \langle$ .

**Notation 5.** The class of all strongly quasi monomorphisms is denoted by  $SQM(\mathcal{X})$ .

Note that  $Mon(\mathcal{X}) \subseteq SQM(\mathcal{X})$ .

**Remark 4.17.** If  $\mathcal{M}$  is a codomain, then it is closed under composition with isomorphisms and it contains  $Mon(\mathcal{X})$ .

**Theorem 4.18.** Suppose that  $\mathcal{M} \subseteq SQM(\mathcal{X})$  and it is a codomain. If  $\mathbf{C}$  is a closure operator such that  $(\mathcal{E}^{QC}, \mathcal{M}^{QC})$  is a quasi factorization structure, then  $\mathbf{C}$  is quasi weakly hereditary and quasi idempotent.

*Proof.* Consider an  $(\mathcal{E}^{QC}, \mathcal{M}^{QC})$  quasi factorization structure,

$$M \xrightarrow{d} N \xrightarrow{n} X$$

of  $m = nd \in \mathcal{M}$ , where  $d \in \mathcal{E}^{QC}$  and  $m \in \mathcal{M}^{QC}$ . So  $m \leq n$  and hence  $\mathbf{c}_X(m) \leq n$ . Since  $\langle m \rangle \subseteq \langle d \rangle$ , we have  $d \in \mathcal{M}$ . Consider the following diagram.

$$\begin{array}{ccc}
 M & \xrightarrow{1_M} & M \\
 \downarrow j_d & & \downarrow j_m \\
 \mathbf{c}_N(M) & \xrightarrow{\quad w \quad} & \mathbf{c}_X(M) \\
 \downarrow \mathbf{c}_N(d) & \quad \quad \quad & \downarrow \mathbf{c}_X(m) \\
 N & \xrightarrow{\quad n \quad} & X
 \end{array}
 \begin{array}{l}
 \text{Left curved arrow: } d \\
 \text{Right curved arrow: } m \\
 \text{Middle horizontal arrow: } \dots\dots\dots w \dots\dots\dots \\
 \text{Bottom horizontal arrow: } n \\
 \text{Vertical arrow from } \mathbf{c}_N(M) \text{ to } N: \mathbf{c}_N(d) \\
 \text{Vertical arrow from } \mathbf{c}_X(M) \text{ to } X: \mathbf{c}_X(m) \\
 \text{Diagonal arrow from } N \text{ to } X: n \\
 \text{Diagonal arrow from } \mathbf{c}_N(M) \text{ to } \mathbf{c}_X(M): w \\
 \text{Diagonal arrow from } N \text{ to } \mathbf{c}_X(M): \text{factorization} \\
 \text{Diagonal arrow from } \mathbf{c}_N(M) \text{ to } N: \text{isomorphism} \\
 \text{Diagonal arrow from } \mathbf{c}_X(M) \text{ to } X: \text{isomorphism}
 \end{array}$$

Since  $d \in \mathcal{E}^{QC} \cap \mathcal{M}$ , by Proposition 3.7 we have  $\mathbf{c}_N(d)$  is an isomorphism. It is easy to see that  $n(d) \leq m$ . Thus  $\mathbf{c}_X(n(d)) \leq \mathbf{c}_X(m)$  and hence  $n(\mathbf{c}_N(d)) \leq \mathbf{c}_X(m)$ . So we have the following commutative triangle,

$$\begin{array}{ccc}
 n(\mathbf{c}_N(M)) & \xrightarrow{\quad w' \quad} & \mathbf{c}_X(M) \\
 \searrow n(\mathbf{c}_N(d)) & \quad \quad \quad & \swarrow \mathbf{c}_X(m) \\
 & X &
 \end{array}$$

where the factorization,

$$\begin{array}{ccc}
 \mathbf{c}_N(M) & \xrightarrow{\quad n\mathbf{c}_N(d) \quad} & X \\
 \searrow e' & \quad \quad \quad & \swarrow n(\mathbf{c}_N(d)) \\
 & n(\mathbf{c}_N(M)) &
 \end{array}$$

is a quasi right  $\mathcal{M}$ -factorization of  $n\mathbf{c}_N(d)$ . Put  $w \stackrel{\text{def}}{=} w'e'$ . So  $n\mathbf{c}_N(d) = \mathbf{c}_X(m)w$ . Since  $\mathbf{c}_N(d)$  is an isomorphism, we have  $n \leq \mathbf{c}_X(m)$ . Therefore  $\mathbf{c}_X(m) \sim n$ , and hence  $\mathbf{c}_X(\mathbf{c}_X(m)) \sim \mathbf{c}_X(m)$ . Thus  $\mathbf{C}$  is quasi idempotent.

Since  $\mathbf{c}_X(m) \leq n$ , there exists  $d' : \mathbf{c}_X(M) \longrightarrow N$  such that  $\mathbf{c}_X(m) = nd'$ . So  $n = nd'w(\mathbf{c}_N(d))^{-1}$  and  $\mathbf{c}_X(m) = \mathbf{c}_X(m)w(\mathbf{c}_N(d))^{-1}d'$ . Thus  $\langle d'w(\mathbf{c}_N(d))^{-1} \rangle = \langle 1_N \rangle$  and  $\langle w(\mathbf{c}_N(d))^{-1}d' \rangle = \langle 1_{\mathbf{c}_X(M)} \rangle$ . These equalities imply that  $w(\mathbf{c}_N(d))^{-1}$  is an isomorphism. It follows that  $w$  is an isomorphism. It is easy to see that  $wj_d \leq j_m$ . Thus by Proposition 2.17 we have  $j_m \in \mathcal{E}^{QC}$ .  $\square$

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